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VOL. 6 BATON ROUGE, LA., NOV-DEC., 1931 No. 3

THE ANGLE-TRISECTION CHIMERA ONCE MORE

We quote in full a clipping mailed to us by one of our graduate mathematical students. It is a proof (?) about which much has been said in recent press notices:

"Pittsburg, Dec. 19, (AP)—The Very Rev. J. J. Callahan, president of DuQuesne University, has offered proof that he has stumped mathematicians 2,000 years.

He announced several months ago that he had split an angle into three equal parts by plane geometry. Scoffing experts said it was 'impossible'. Now Father Callahan comes forward with a figure and explanation of his method which he describes as quite 'simple'. Here it is:

'Let AC and DF be two parallel lines at any given distance apart.

'On one parallel line take a certain distance DF.

'With D as a center and DF as a radius, intersect AC at C, making DC equal to DF.

'With F as a center and a radius equal to DF intersect AC at A, making AF equal to DF.

'Join AF and DC. On DC as a base, construct the angle DCE equal to angle ACD. From D draw DB parallel to EC and DE parallel to AF.

'Then DC and DF trisect the angle BDE."

Before we proceed to point out the mistake made by President Callahan when he offers the above construction to the world as one which mathematicians for 3,000 years had struggled vainly to discover, it will be proper and, doubtless, for many readers of the NEWS LETTER, instructive, to recall something of the historic setting of the problem and to state its precise nature.

The problem arose among the ancient Greeks whose system of geometry originated largely with their own countrymen, Euclid. The elements of Euclid's geometry may be said to have been restricted to straight lines, circles, angles and combinations of them. Thus, correspondingly, every construction of these elements or combinations of them was possible by the use of one or both of the instruments, the straight edge and a pair of compasses, and these only. Hence, the classic problem of angle-trisection is that problem, and *only* that one, which seeks the division of an arbitrary plane angle into three equal parts by the use *only* of the *straight edge* and *compasses*, that is, by using the *straight line* and the circle.

It may be remarked in passing that nothing is of greater importance in the art (or science?) of problem-solving than a knowledge that the identity of every problem is determined only when the following three things have been previously determined, namely, (a), the precise hypotheses, (b), the objective to be attained, (c), the particular mode, if one should be prescribed, by which the objective is to be attained. The importance of these distinctions is exemplified in the fact that many would-be solvers of the classic trisection problem have been deceived into thinking they had solved it by overlooking, or being ignorant of, their use of some type of motion distinct from the circular motion of compasses, for instance, by a lateral sliding motion of the straight edge.

In this connection it is appropriate to cite a few different kinds of the general angle-trisection problem.

Problem I. Given an arbitrary plane angle A, to divide it into three equal angles, the mode of division being restricted to the use of the straight edge, the circle, the hyperbola.

This problem was, according to Ball, the mathematical historian, first solved by Viviani, an Italian mathematician, in 1677. A simple and interesting explanation of how it is done was published by Professor H. E. Buchanan in Vol. 3, No. 4, of the Mathematics News Letter, the title of his article being "The Development of Elementary Geometry".

Problem II. Given an arbitrary plane angle A, to trisect it, the mode of division being restricted to use of the straight edge, the circle, and the quadratrix. (The quadratrix is the locus of points of intersection of a straight line made to move parallel to itself in the plane while constantly interesecting the radius which describes a circle, the two motions being uniform, and so timed that the straight line moves through the length of the radius while the radius is describing an angle of 90 degrees.)

This problem, according to the historians, was first solved by Dinostratus, a pupil of Eudoxus, some time during the fourth century B. C.

Problem III. Given an arbitrary angle A, to trisect it, the mode of division being restricted to use of the straight edge, the circle, and the curve (conchoid) defined by the polar equation $r = a \sec \theta \pm d$.

This problem was also solved by Viviani in 1677.

Problem IV. (The Classic Problem). Given an arbitrary angle A, to divide it into three equal parts, the mode of division being restricted to use of straight edge and compasses.

Proofs of the impossibility of the solution of this problem are now parts of classic mathematical literature. One desiring a simple elementary proof of such impossibility will find it in Dickson's First Course in the Theory of Equations. It consists essentially in showing that only rational numbers and numbers expressible in the form of a finite number of quadratic irrationals are constructible by ruler and compasses. Since the equations of a straight line and a circle are of the first and second degree respectively when referred to a pair of rectangular axes (and they can *always* be so referred) no numbers other than the kind just described can result from the simultation of such equations. Since the third part of a given angle cannot in general be related to the whole angle by a quadratic or a linear relation, it follows that an arbitrary angle cannot be trisected.

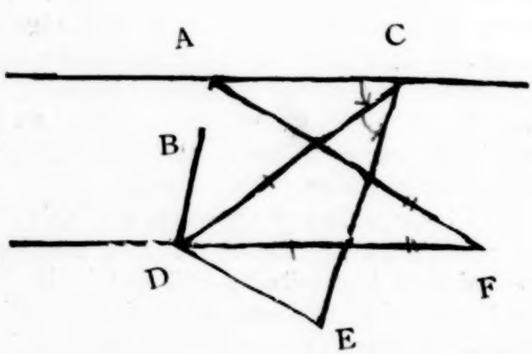
Such citations from the modern literature as the following are so common that their number can scarcely be estimated.

"Whenever you see in the newspapers a statement that some budding mathematician, a Sidis, for example has achieved immortal fame by trisecting an angle you may at once put it down as mere bunk. It cannot be done with ruler and compasses and it has been done half a dozen different ways more than 2000 years ago by the Greeks themselves using special curves."—H. E. Buchanan."

"To this day there are professional 'trisectors' whose greatest handicap lies in the fact that they have never learned that the problem was liquidated three hundred years ago."—Tobias Dantzig, in "Number the Language of Science."

"Like the problem of the duplication of a cube, the trisection of an angle cannot be solved by the methods of elementary geometry."—Simon Newcomb.

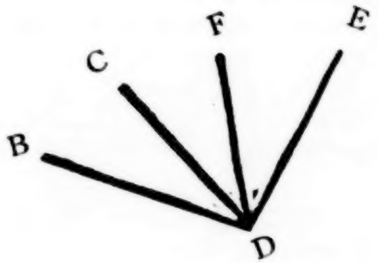
We now revert to our original task of pointing out the nature of Father Callahan's error in his trisection (?) argument. The reader will better understand what follows if a figure be drawn showing the constructions.



Instead of solving the problem: given an arbitrary angle BDE, to find by ruler and straight edge an angle, as BDC, which shall be one-third of angle BDE, he has *only solved* what indeed is a "quite simple" problem, namely, given an angle DCE, to find by ruler and compasses a second angle, as BDE, which shall be three times as large as DCE.

Put otherwise, instead of dividing a given angle into three equal parts by ruler and compasses, he has merely *multiplied* a given angle three times, an extremely simple, almost trivial, construction—one

which could have been done in a still far more simple way, by merely placing in consecutively adjacent order three equal angles, as shown in the following figure.



Plainly, DC and DF divide angle BDE into three equal parts, BUT, angle BDE WAS NOT the GIVEN ANGLE, as the classic problem requires. In other words, instead of laying down the angle BDE *first*, then operating upon it with a straight edge and compasses in such a manner as finally to arrive at an angle BDC which should be BDE/3 the Callahan construction begins with an angle, as BDC, and, operating upon that angle, finally arrives at angle BDE. If the former could have been done, it would, of course, have been a solution of the classic problem. But TO DO THE FORMER IS IMPOSSIBLE, as has already been repeatedly indicated.

In closing this discussion, we quote once more from that very inspiring, very stimulating book by Tobias Dantzig, "Number, the Language of Science". "Yes! The impossibility of the classical problem was imposed by a restriction which was so old as to be considered natural, so natural, indeed, that it was rarely mentioned. When the Greek spoke of a geometrical construction, he meant a construction by straight edge and compasses. These were the instruments of the gods; all others were banned as unworthy of the speculation of the philosopher. For Greek philosophy, we must remember, was essentially aristocratic. The methods of the artisan, ingenious and elegant though they may have been, were regarded as vulgar and banal, and general contempt attached to all those who used their knowledge for gainful ends . . . . The trouble with the cubic equations to which the doubling of the cube and the trisection of the general angle lead is that they are irreducible (to

lower degree), and this fact condemns the problems which are behind these equations as insoluble by ruler and compass.

We have here another confirmation of the relative nature of the term impossible. Impossibility is nearly always the result of a restriction, usually a restriction so sanctified by tradition that it seems imposed by nature itself. Remove the restriction and the impossibility will disappear."—S. T. S.

PUPIL ATTITUDE TO MATHEMATICS

Psychologists tell us that the student who makes A's and B's in some subjects should make them in all others. Probably he should but the fact is that he often does not. It seems that this drop from the average grade is more prevalent in mathematics than in any other subject. No doubt there are a number of immediate causes for this, but each cause may be traced back to one of the two following conditions:

Often the student is trying to handle material that is too difficult for him. This may mean that he is not prepared for the particular course in mathematics which he is taking, but in high school, it usually means that he has gotten lost in that particular course, for Algebra and Geometry do not have the decided prerequisites that college mathematics has. Of course the student who has been absent or otherwise hindered in his work gets behind and the teacher should, and usually does, help him to catch up, but we often let students, even the most alert ones, lose out before our very eyes without realizing it. A student may "miss the point", may fail to grasp the full meaning, may not see the true relationship that exists, and still may be able to solve his problems in a parrotlike way. When this is the case, he soon forgets the system and is not able to transfer it to other situations. For instance, I have seen many college students who were unable to recall synthetic division. They had forgotten it because they had learned it as a form, failing to see the relation between it and long division, that it is only a shortening of what they have always done. I have seen students stumble over problems because they were unable to make use of the discriminant. They had derived it in high school, no doubt, but had failed to get its true meaning. If a student is allowed to continue from day to day without grasping the full meaning of it, without knowing and understanding what he is about, and

without seeing the truth and beauty of mathematics, he will certainly lose interest, and join the multitude who claim that mathematics is dull, dry and uninteresting, and that he will never need it. Therefore, it behooves the teachers of high school mathematics to be very explicit of both details and principles. Too many teachers are inclined to pass over very lightly many things, because they are "so easy", not realizing that what is easy for them may be complicated to the student. Of course the teacher has so much ground to cover that she has no time to waste, therefore, she must know her subject thoroughly, select her material wisely and present it in a definite, clear, to-the-point way.

Many students are not interested in mathematics as a science. All they care to know about it is only the essentials that will probably be needed in every day life. They always ask the question, "What is the use of all this?" "When will I ever need this?" etc. The teacher knows in the bottom of her heart that most likely they will never need it, but she must teach it just the same. This attitude on the part of the student is not always to be condemned or changed, for each student should be allowed complete freedom in selecting his vocation or profession and more mathematics than he is going to need should not be forced upon him. Other students claim a lack of talent for mathematics. While no doubt the real mathematician, has inherited a make-up somewhat different from that of the poet, the musician, or the aritist, the ability to master a fair amount of high school mathematics is certainly possessed by every normal person.

It is usually in the high school that the permanent attitude toward mathematics is formed. The teacher has a wonderful opportunity here. It is up to her to arouse a real live interest and spirit and to instill into her students a genuine love and respect for this wonderful and beautiful science.—A. H. S.

A UNIT IN FIELD MATHEMATICS

By GERALDINE McCALL
Supervisor of Mathematics
Mississippi Industrial & Training School.

One of the differences between high school mathematics as taught fifteen or twenty years ago and mathematics as we strive to teach it today is a change of emphasis from the development of mere mechanical

skills to a more realistic understanding of the processes and problems involved. As evidence of this change in present day mathematics, we notice recent changes in courses of study. For instance, in algebra the fundamentals of elementary trigonometry are introduced, and the function concept and work in graphs are extended. Mathematics teachers should not be so absorbed in teaching symbols and manipulations that they neglect to point out the meaning of the principles involved and their uses in deriving practical rules of greater utility.

The two processes which caused mathematics to come into existence, at the dawn of human history, were counting and measuring. Mathematics has been defined as the science of measurement. This definition, while very incomplete, may serve to remind us of its origin.

In view of the change in ideas and the wide spread dissatisfaction with present mathematics courses on the part of the general public, we have entered a unit of Surveying, or Field Mathematics, in our course of study at the Mississippi Industrial and Training School, in all classes in Junior and Senior High School Mathematics. Our aim was to construct a course which should give to pupils the sort of mathematics best adapted to their ages, interests, and intellectual endowments.

All teachers of mathematics know that boys, in particular, pass through a stage of development which might be called the "civil engineering" stage. During this period of development, the interest in engineering is real and teachers of mathematics can motivate their work by making use of these interests. It is a wonderful change to get away from the text and out in the woods, where the children can make a map of the surrounding country showing definite location of "hidden treasure," find the height of a tall tree with a crow's nest in it, measure the distance across a pond, or the height of a mound.

Particularly in classes where we do not always secure sufficient interest and enthusiasm, we cannot afford to neglect any appeal to interest. This interest appeal always serves to motivate and vitalize the pupil's work.

Much of the work in Field Mathematics is accomplished with very simple instruments. Many of these are easy to construct, and all of them may be bought at a reasonable price from the Lafayette Instrument Co., New York. The instruments are easy to use, requiring only a knowledge of geometrical principles, and are very interesting.

The angle mirror and sextant work in the same way. They are easy to manipulate and they have some interesting geometry as a foundation for their construction and their use. The hypsometer and clinometer is a modern form of the quadrant, one of the most popular instruments of the middle ages. It may be used for finding angles of elevation or depression; for finding sines, cosines, and tangents; for making indirect horizontal measurements; and for leveling.

The geography teacher may make use of the angle mirror to show pupils how to make maps. In geography, too, the pupil comes in contact with degrees, minutes, and seconds. If he is given some instrument like the hypsometer and clinometer with which he can measure angles he will get a much clearer concept of the meaning of angles and a fuller appreciation of the practical value of such instruments.

The transit, the most important of all measuring instruments, is simply a field protractor for measuring horizontal, and, usually, vertical angles. One may be bought for from $20 to $40 from the instrument company already named, or one can be easily made with a large protractor, two inexpensive carpenters levels, two rulers, and a plumb line. The transit will do the work of the other instruments already described, but greater interest is aroused by a variety of instruments.

The plane table is one of the most important instruments for making general maps. It is so simple in construction that a seventh grade child may learn to use it in a few minutes and so accurate that many of the best engineers prefer it to any other instrument for making surveys and maps. With the plane table we use a sighting instrument called an alidade. A simple homemade outfit may be made with a drawing board on a tripod, and a small box with slits cut in the ends for an alidade. With the plane table, all the important work of the map is done in the field. Angles are drawn directly on the map and not measured and then plotted.

In the beginning of this course in Field Mathematics the pupil should be shown that he can make certain measurements directly by using a ruler, or a steel tape, but that many measurements, such as the distance across a wide lake, the height of a mound, or the distance to the moon, must be made indirectly. It is well to call the pupil's attention to the fact that one way of making an indirect measurement is to measure enough lines related to the required distance to enable

one to draw the figure to scale, after which the required distance may be computed from the drawing.

Pupils must be impressed with the fact that all data which have been found by measuring are only approximate, and that all measurements on any piece of work should be made with the same degree of accuracy. The most important and fundamental work in Surveying, or Field Mathematics, is accurate linear measurement. Ask your pupils to measure the length of the top of a desk, and note the different answers given.

William Betz says in an article on the *Teaching of Direct Measurement in the Junior High School*: ". . . the best results seem to be obtained if the work in direct measurement is carried on in four distinct, successive steps, as follows:

(1) There should be careful study of the measuring instrument.
(2) There must be adequate practice in prescribed measurement.
(3) Definite provision must be made for the development of accuracy, through "controlled" measurement.
(4) The skill thus acquired should then be exercised extensively, through suitable application."

We have at Mississippi Industrial and Training School many of the instruments already described. Each child is supplied with a note book, comparative scale ruler, protractor, compasses, and colored pencils. We have a mathematics laboratory in which the maps are drawn by the pupils during their vacant periods, under supervision of the mathematics teacher. In the Junior High School the work is done on the field as in the other classes, but the maps are done by Scale Drawing. The geometry class does a greater part of this unit in Field Mathematics by use of congruent and similar triangles.

In the first year algebra class the children are introduced to trigonometry soon after the study of simple equations. The tangent, cosine, and sine tables are represented graphically so as to give the child a practical insight into the real meaning of these tables. After a study of the theoretical use of these tables they were used in working out some practical problems here on the campus. By use of the tangent table the following maps were made: A map of our swimming pool, showing its length and width, (then by arithmetic the area was found); a map showing how the height of a radio pole was computed; and finally a map showing how the distance of a ship out in the water was found. By use of a cosine table we found the length of a guy

wire attached to a leaning telelphone pole; computed an angle formed by a ladder leaning against a house; and found the distance from a gun at a point A to a battery at point B. In the latter problem imaginary objects were used. By use of the sine law we found the height of a mound; the height of a balloon; and the number of feet one would rise in walking one hundred feet up a railroad track if the angle of elevation was twenty-seven degrees.

In an effort to further arouse the interest of the children, a problem closely relating to the glamorous hectic stories of Captain Kidd and his bold pirates was devised. In piracy lore the word "chart" or "map" is invariably connected with a faded sketch of a lonely island, dead trees, and buried treasure. One section of the first year algebra class buried the "treasure" and then plotted and mapped the ground even as pirates of old did. The other section of the class took the sketching and figured out the location of the "hidden treasure."

The last maps made by the students are practical drawings of the areas of different pieces of land on our campus, and on the farm.

Mathematics can be taught without instruments, but it can be taught much better with them. Thousands of dollars worth of science apparatus is bought in our schools each year, and the mathematics equipment usually consists of several old broken yardsticks. The saving in teacher cost and failures each year would more than pay the cost of the instruments. Many constructions, problems, and propositions that formerly seemed dry and useless take on new meaning as a result of this practical field work. The mere presence of the instruments in a mathematics room develops interests which last throughout the course.

After completing the work in Field Mathematics at the Industrial and Training School the children have come to feel that mathematics is a vital part of life, and they can already imagine themselves engineers applying mathematics to real life situations. Field Mathematics has indeed proven to be a powerful motivating force in our school.

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ON THE SINE FUNCTION

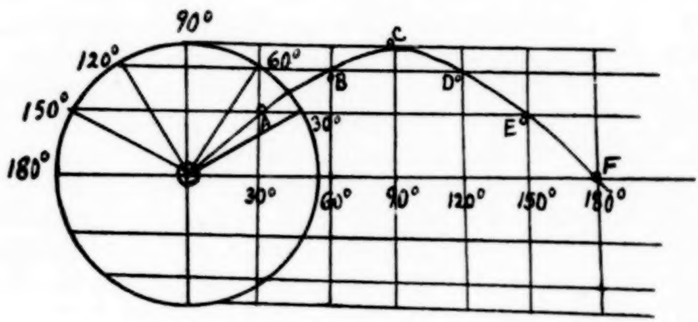
By W. PAUL WEBBER
Louisiana State University.

When we say the slope of a straight line joining two points (rectangular coordinates) is the difference of the ordinates divided by the difference of the abscissas, it is equivalent to saying the slope is the change in the ordinate per unit change of abscissa. The slope of a curve at a point is defined as the slope of its tangent at that point. Ordinarily, when we construct a curve we use the same units of length for measurements in all directions. This is the fundamental notion of a map of which the ordinary graph is a particular type. Any deviation from this usage is to be justified by some special reason, for example, to deliberately enlarge irregularities in one direction or in order to plot large numbers on a small sheet of paper, or some similar reason. It is only when a graph representing space relations or a map is drawn to the same scale in all directions, that we obtain what may be termed the natural representation of the thing mapped. Of course analytically, this is not necessary, for the reason that pictures are used conventionally in analysis.

With these ideas in mind let us examine the sine function of trigonometry as the equation $y = \sin x$. To get the back ground we take the original definition of the sine as given by the Hindus. The sine of an arc (measure of angle at center of circle) is half the chord of twice that arc. This definition plainly implies that the arc and chord and radius are to be measured by the same unit. If $\Pi = 3.1416$ is the ratio of the circumference to the diameter the diameter (and radius) and circumference are measured by the same unit. It seems that historically all measurements relating to the circle are based on this idea.

To picture to the eye the variation of the sine of an angle as the angle varies say from 0° to 180° we recall that this will correspond to the motion of a point on the circumference over the semi-circum-

ference. To do this according to the historical definition the arcs corresponding to the values of the angle must be laid off on the axis of abscissas. It is natural to make its scale to the same unit as employed to measure the radius. This implies the radius as a common unit. Then to lay off an arc measuring 30° at the center we take a section on the axis equal to one twelfth of the circumference. This is illustrated in the diagram below. This diagram may be found in most any text on physics or elementary trigonometry:



This gives what may be called (in the light of what we have said above) the natural form and proportions of the curve. These have been based on purely geometric considerations.

Now consider the slope of the curve represented by $y = \sin x$ at $x = 0$, for example. It will be seen from what has been said that the radius is suggested as a convenient unit and the angle subtended by an arc equal to the radius as a unit angle. This is implied in the above discussion. Now we have

$$\left.\frac{d}{dx}\sin x\right]_{x=0} = \left.\frac{\sin x}{x}\right]_{x=0} = 1$$

as is proved in any text book. Thus it appears that the geometric analysis leads to the customary result of calculus in a historical and natural way.

ON CERTAIN DEFICIENCIES OF COLLEGE FRESHMEN IN ALGEBRA

By S. B. MURRAY
A. & M. College, Mississippi

Aside from the rather apparent fact that fully half of an average Freshman class exhibits no more than a hazy idea of mathematics in general and of algebra in particular, a marked deficiency in albegra seems to be a disregard of signs. Whether a term has a positive or a negative sign causes the average beginner in college algebra but little concern. To the student of less than average ability, it causes no concern. That mistakes in the proper use of signs are not all caused by carelessness is shown by the number of students who, after deliberation, are undecided as to what to do in operations involving changes in signs. Usually at least one out of twenty men will ask a question about proper use of signs at each meeting of the section. This is by no means confined to Freshmen sections in algebra. It has occurred upon more than one occasion in Sophomore sections in the Calculus. Asking the question is, of course, the thing to do, but the fact seems to indicate a chronic doubt as to the correct use of signs.

That a term may carry its own sign is an idea which half of the students entering college algebra have not grasped. Sometimes some students, in multiplying polynomial expressions, will bring down the products first and then go back and supply the signs. The result sometimes suggests that the idea was to place signs so that the result would present the best general appearance. The need of well planned drills is obvious.

Simple cases of factoring are, on the whole, fairly well handled. More involved and special cases of factoring are seldom solved except by the best students. Such types as the sum of difference of two cubes, the difference of two squares where one of the squares is a polynomial expression, or any type of the same or greater difficulty is beyond the knowledge or ability of three-fourths of the Freshmen entering college algebra.

It has been the writer's observation that fractions give about as much trouble as anything in high school algebra, not excepting radicals and exponents. At least 80% of the Freshmen entering college algebra have difficulty with fractions. Half of the 80% experience considerable difficulty with fractions, and such trouble persists even into the Calculus. Such a deficiency makes the latter course much harder

than would ordinarily be the case. A college course cannot dwell at length on these deficiencies and cover the material necessary for advancement. Possibly addition or subtraction of fractions with polynomial numerators and denominators give the greatest trouble, though simplification of complex fractions seems almost an impossibility for the lower third of students entering college algebra.

Operations with exponents and radicals are basic in the treatment of series, logarithms, partial fractions, and the solution of equations. Yet we find a large per cent who show little knowledge of the rules by which such expressions are combined or transformed. Even less appears as regards ability to operate expressions which involve simple exponents or radicals. We find in many cases little or no knowledge of praghic representation of simple equations. The ability to read a mathematical statement is sadly lacking. The interpretation of simple problem situations is a point of weakness which reveals this condition and even the simplest paragraphs seem to leave many in a confused state of mind. The essential words and phrases seem not to belong to the student. We are confronted with a lack of mental integrity. By this I mean that many seem to think that if they can copy something from a neighbor which they do not understand they have done all that is necessary.

These observations are made in the hope of suggesting points of emphasis in high school algebra. The remedy is obvious. Adequate drill in fundamentals is the burden of the high school, and students should not be promoted to college until a fair showing is made on these topics.

NOTE ON THE INTEGRATION OF A RATIONAL FUNCTION OF TRIGONOMETRIC FUNCTIONS AND QUADRATIC RADICALS

By H. L. SMITH
Louisiana State University.

1. Integration of R $(x, \sqrt{a+bx+cx^2})dx$. A number of years ago while teaching section 197 of Granville's* Calculus to a class of sophomores, it suddenly occurred to me that the method of that section could be used to integrate expressions of the form

$$R(x, \sqrt{a+bx+cx^2})dx,$$

*Granville, Elements of the Differential and Integral Calculus, Revised Edition.

where R is a rational function. I, therefore, on the spur of the moment, suggested to the class that they re-solve the exercise on p. 337 by this method, the detailed directions being equivalent to the following:

(1) Reduce $R(x, \sqrt{a+bx+cx^2})dx$ to the form

$$R_1\,(y, \sqrt{A+Cy^2})dy$$

by means of the substitution

$$y=x+b/2c;$$

(2) Reduce $R_1\,(y, \sqrt{A+Cy^2})dy$ to the form

$R_2\,(\sin\theta, \cos\theta)d\theta$ by means of an appropriate substitution of one of the forms

$$y=r\sin\theta\ , y=r\tan\theta,\ y=r\sec\theta;$$

(3) Reduce $R_2\,(\sin\theta, \cos\theta)\,d\theta$ to the form

$R_3\,(z)dz$, by means of the substitution

$$z=\tan\theta/_2$$

This procedure had the obvious advantage of being easy to remember and I did not foresee any difficulty with it. But when the class brought in their work the next day not one of them had obtained a single one of the answers given by Granville on p. 339. All of the answers obtained were more complicated than the corresponding ones given by Granville and could be reduced to his only with considerable difficulty. This struck me as odd, and I sought the reason for it. I found that if in the steps (2), (3) the function $\sin\theta, \tan\theta, \sec\theta, \tan\theta/_2$ were replaced by the corresponding co-functions, then Granville's answers were obtained at once. The reason for this was clear on noting that in the revised method the transformation of the answer from an expression in z to an expression in x was easily carried out by aid of the identity.

$$\operatorname{ctn}\theta/_2=\csc\theta+\operatorname{ctn}\theta,$$

but was not so simply carried out in the original method where the identity to be used was

$$\tan\theta/_2=\csc\theta-\operatorname{ctn}\theta.$$

2. Integration of R (sin θ, cos θ)dθ. The integration of R(sin θ, cos θ)dθ is usually effected by means of the substitutions

$$z = \tan \theta/_2 = \csc \theta - \operatorname{ctn} \theta$$

or the substitution

$$z = \operatorname{ctn} \theta/_2 = \csc \theta + \operatorname{ctn} \theta.$$

The actual expressions for sin θ, cos θ, dθ in terms of z are, however, hard to remember. To avoid this burden on the memory the following method is suggested.

We note first that R(tan θ)dθ can be integrated by writing it in the form

$[R(\tan \theta)/(1+\tan^2 \theta)] \sec^2 \theta \, d\theta$ and putting $y = \tan \theta$. Next we note that R(sin θ, cos θ) dθ can be written in the form $R_1(\tan \theta)d\theta$ if R (sin θ, cos θ) is a fraction whose numerator and denominator are both homogeneous in sin θ, cos θ and of the same degree. For we have only to multiply both numerator and denominator by the appropriate power of sec θ. We next note that the numerator and denominator of R(sin θ, cos θ) can be made homogeneous and of the same degree, provided each term is of *even* degree. For then we have only to multipiy each term by a properly chosen power of $(\cos^2\theta + \sin^2\theta)$. Finally we note that R(sin θ, cos θ)dθ can always be written in the form R_2 (sin $\theta/_2$, cos $\theta/_2$)dθ, where each term of the numerator and each term of the denominator is of even degree. To do this we have only to replace sin θ and cos θ by $2\sin \theta/_2 \cos \theta/_2$, $\cos^2\theta/_2 - \sin^2\theta/_2$, respectively.

ON TANGENTS TO CURVES OF EVEN DEGREE

By W. V. PARKER
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In a recent paper (Bulletin of the American Mathematical Society, August, 1931) I gave a method for getting the curves of the type $Y = f_k(x), \frac{1}{2}n \leq k \leq n$, which are tangent to $y^2 = f_n(x)$ at all points where the two curves meet, where the functions are polynominals of degree indicated by their subscripts. In the present paper I have considered some of the special features of the case when n is even and $k = \frac{1}{2}n$, and have given special attention to the case $n = 4$.

The applications of the theorem of the above mentioned paper to conics are, I believe, well known but it might be well to recall them here. Suppose that it is desired to construct a tangent to an ellipse at any point P on it. Join P to the extremities of either axis by lines q_1 and q_2 and draw any line p, not through P, perpendicular to this axis. If M is the mid-point of the segment cut off on p by q_1 and q_2, then the line joining M to P is tangent to the ellipse at P. The construction is the same for the hyperbola except that P must be joined to the extremities of the transverse axis. For the parabola one of the lines joins P to the vertex and the other is parallel to the axis, the rest of the construction is the same.

The case $n=4$. Let C_4 be the curve $y^2=f_4(x)$, c_1 the curve $y=f_2(x)$ and c_2 the curve $y=g_2(x)$, then the curve S whose equation is $y=\frac{1}{2}[f_2(x)+g_2(x)]$ is tangent to C_4 at the two points where c_1 and c_2 meet if and only if $f_2(x)g_2(x)\equiv f_4(x)$. Write $f_4(x)\equiv a(x-e_1)(x-e_2)(x-e_3)(x-e_4)$ then if $f_2(x)\equiv\sqrt{-a}(x-e_i)(x-e_j)$ and $g_2(x)\equiv-\sqrt{-a}(x-e_k)(x-e_l)$, where (i, j, k, l) is some permutation of (1, 2, 3, 4), the above condition is satisfied. In this case S is a straight line and hence is a bitangent* of the quartic curve C_4. *Conversely, all bitangents of* C_4 *may be obtained by this process.* Suppose there is a line L tangent to C_4 at each of two points (u_1, v_1); (u_2, v_2) and let this equation be $y=q(x)$. The equation $f_4(x)-q^2(x)=0$ has the roots u_1, u_2 each counted twice. Let s be the curve of the form $y=h_2(x)$ determined by (u_1, v_1); (u_2, v_2) and $(e_i,0)$. Since L and s meet in (u_1, v_1); (u_2, v_2), the equation $h_2(x)-q(x)=0$ has roots u_1, u_2. We have, therefore,

$$[h_2(x)-q(x)]^2\equiv\mu[f_4(x)-q^2(x)],$$

and hence

$$[h_2(e_i)-q(e_i)]^2=\mu[f_4(e_i)-q^2(e_i)];$$

but $h_2(e_i)=f_4(e_i)=0$, hence $\mu=-1$, and we have

$$h^2_2(x)-2h_2(x)q(x)+q^2(x)\equiv q^2(x)-f_4(x),$$

or

$$f_4(x)\equiv h_2(x)\,[2q(x)-h_2(x)].$$

We see therefore that $h_2(x)$ is a factor of $f_4(x)$. If now we write $2q(x)-h_2(x)\equiv k_2(x)$, we have $f_4(x)\equiv h_2(x)k_2(x)$, but $\frac{1}{2}[h_2(x)+k_2(x)]\equiv l(x)$ that is the sum of $h_2(x)$ and $k_2(x)$ is linear and hence they must be

*The term bitangent as used here refers to a line having two point contact with C_4 at each of two points. In certain special cases these points may coincide in which case the line has four point contact with C_4 at one point.

contained among the functions $f_2(x)$ and $g_2(x)$ above. We have, therefore, the following theorem.

Theorem 1. *If the zeros of $f_4(x)$ are real, a necessary and sufficient condition that C_4 have a real bitangent is that* a *be negative.*

In case the zeros of $f_4(x)$ are real and distinct the curve C_4 has two loops, one finite and one infinite or both finite. The latter case is the one for which we have real bitangents. In this case there are six real bitangents. Of these six two are tangent at imaginary points. In particular if $e_i+e_j=e_k+e_l$ there is a bitangent $y=c$ parallel to the x-axis and the curve $y_2=\frac{1}{2}[\lambda f_4(x)+C]$ will have four point contact with C_4 at the two points of tangency of this bitangent provided that λ is so chosen that the curve $y=\lambda f_4(x)$ goes through one of these points.

The case $n=2k$. Let C_{2k} be the curve $y^2=f_{2k}(x)$, c_1 the curve $y=f_k(x)$ and c_2 the curve $y=g_k(x)$, then the curve S whose equation is $y=\frac{1}{2}[f_k(x)+g_k(x)]$ is tangent to C_{2k} at each of the k points of intersection of c_1 and c_2 if and only if $f_k(x)g_k(x)\equiv f_{2k}(x)$. Write $f_{2k}(x)\equiv a(x-e_1)\ (x-e_2)\ .\ .\ .\ .\ (x-e_{2k})$ then if $f_k(x)\equiv\sqrt{-a}(x-e_{i_1})\ .\ .\ .\ (x-e_{i_k})$ and $g_k(x)\equiv-\sqrt{-a}(x-e_{i_{k+1}})(x-e_{i_{k+2}})\ .\ .\ .\ .\ (x-e_{i_{2k}})$, where $(i_1, i_2, \ldots, i_{2k})$ is some permutation of $(1, 2, \ldots 2k)$, the above condition is satisfied. In this case the degree of S is k-l or less. Let $E_1, E_2, \ldots, E_k$ be the symmetric functions of the roots of $f_k(x)=0$, and let $H_1, H_2, \ldots, H_k$ be the symmetric functions of the roots of $g_k(x)=0$. Then if $E_1=H_1$, $E_2=H_2, \ldots, E_{l-1}=H_{l-1}$ but $E_l\neq H_l$, the degree of S will be $k-q$. Conversely, any curve L of the form $y=\varphi_{k-1}(x)$ which is tangent to C_{2k} at each of k points may be obtained by this process. Denote the K points of tangency of L with C_{2k} by (u_1, v_1); (u_2, v_2); . . . . ; (u_k, v_k). Let s be the curve of the form $y=\psi_k(x)$ determined by these k points together with $(e_i, 0)$. The roots of the equation $\psi_k(x)-\varphi_{k-1}(x)=0$ are $u_1, u_2, \ldots, u_k$, and the roots of the equation $f_{2k}(x)-\varphi^2_{k-1}(x)=0$ are $u_1, u_2, \ldots, u_k$ each counted twice. We have, therefore,

$$f_{2k}(x)-\varphi^2_{k-1}(x)\equiv\mu[\psi_k(x)-\varphi_{k-1}(x)]^2,$$

and hence

$$f_{2k}(e)_i-\varphi^2_{k-1}(e_i)=\mu[\psi_k(e_i)-\varphi_{k-1}(e_i)]^2;$$

but $f_{2k}(e_i)=\psi_k(e_i)=0$, hence $\mu=-1$, and we have

$$f_{2k}(x)\equiv\psi_k(x)[2\varphi_{k-1}(x)-\psi_k(x)].$$

We see that $\psi_k(x)$ is a factor of $f_{2k}(x)$. If now we write $2\varphi_{k-1}(x) - \psi_k(x) \equiv \eta_k(x)$, we have $\psi_k(x)\eta_k(x) \equiv f_{2k}(x)$, but $\frac{1}{2}[\psi_k(x) + \eta_k(x)] = \varphi_{k-1}(x)$ and hence $\psi_k(x)$ and $\eta_k(x)$ must be contained among the functions of $f_k(x)$ and $g_k(x)$ above. We have, therefore the following Theorem:

Theorem 2. *If the zeros of $f_{2k}(x)$ are real, a necessary and sufficient condition that there be a real curve of the form $y = f_{k-1}(x)$, $q > 0$, which is tangent to C_{2k} at each of k points* is that a be negative and that $E_1 = H_1$, $E_2 = H_2$, ... ,$E_{l-1} = H_{l-1}$, and $E_l \neq H_l$ for some choice of $f_k(x)$ and $g_k(x)$.*

In particular if a is negative and $E_1 = H_1$, $E_2 = H_2$, ... ,$E_{k-2} = H_{k-2}$, there is a line tangent to C_{2k} at each of k points. If in addition $E_{k-1} = H_{k-1}$ but $E_k \neq H_k$ for some choice of the functions $f_k(x)$ and $g_k(x)$, there will be such a tangent line $y = C$ parallel to the x-axis. The curve $y = \frac{1}{2}[\lambda f_{2k}(x) + c]$ will have four point contact with C_{2k} at each of the k points of tangency of this line provided that λ is so chosen that the curve $y = \lambda f_{2k}(x)$ goes through one of these points. The value of λ satisfying this condition is $\lambda = \dfrac{1}{c}$

*These points may not be all different. If p points coincide at a point this point counts as 2p among the interesections of the curves.

MATHEMATICS AT A. & M. COLLEGE, MISSISSIPPI

By C. D. SMITH

The Mathematics Club at Mississippi Agricultural and Mechanical College, together with the Club at Mississippi State College for Women felt that a service could be rendered to undergraduates by forming a national honorary fraternity in mathematics. After consideration of the proposed fraternity the two clubs appointed a committee to draw up a constitution and to take the necessary steps to obtain a charter for the newly organized fraternity.

The main purpose of the fraternity is to stimulate interest in mathematics, promote scholarship, and foster fellowship among undergraduate students in four year colleges and universities.

The fraternity proposes to realize these ends by endeavoring to assemble a group of select students who have similar interests and

aspirations. This is done by selection since only students who have shown ability in their work and interest in advanced work in mathematics are eligible.

Since Mathematics Clubs are in existence in almost all of the universities and colleges and as their work is similar to that of the fraternity they will be invited to apply for a charter, thus giving additional values to club members in that the fraternity is national in scope. Such an organization of clubs will give a condition of permanence and wider fellowship which cannot be had in a club which serves only local interests. We would hope to establish in this way a close cooperation among the groups of different schools. The possibilities and advantages of the organization are obvious. The widespread interest in local clubs suggests the need of affiliations and we hope to be able to present the matter to the clubs in this section at an early date.

PROBLEM DEPARTMENT

Edited by
T. A. BICKERSTAFF
University of Mississippi

This department aims to provide problems of varying degrees of difficulty which will interest any one engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

Problems for Solution

No. 12. Proposed by the Editor:

A pendulum swings east and west in a wind from the west so that each swing westward is 7/10 as long as the preceding eastward swing and each swing eastward is 13/10 as long as the preceding westward swing. Find the distance traversed by the pendulum in coming to rest if the first swing is eastward and equals 100 centimeters, assuming that the time of each swing is proportional to the length.

No. 13. Proposed by W. Vann Parker, Mississippi Woman's College, Hattiesburg, Mississippi.

A goat is staked on the edge of a circular grass plot of radius R with a rope of Length L. What must be the value of L in terms of R so that the goat can graze over one-half of the plot?

No. 14. Proposed by the Editor:

When I was born, my sister was $\frac{1}{4}$ as old as my mother. She is now $\frac{1}{3}$ as old as father. In four years, I shall be $\frac{1}{4}$ as old as father. I am now $\frac{1}{4}$ as old as mother. How old is each member of our family?

Solutions of Problems

No. 4. Proposed by the Editor. Solved by Chester R. Hillard, Wheaton College, Wheaton, Illinois:

Express the square root of $18+2\sqrt{14}+6\sqrt{2}+2\sqrt{63}$

as the sum of three square roots.

Solution:

Let, $\sqrt{18+2\sqrt{14}+6\sqrt{2}+2\sqrt{63}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$

Then, $18+2\sqrt{14}+6\sqrt{2}+2\sqrt{63}=x+y+z+2\sqrt{19}+2\sqrt{xz}+2\sqrt{yz}$

xy

Now if, $x+y+z=18$

$xy=14$

$xz=18$

$yz=63$

We get, $x=2$, $y=7$, and $z=9$.

Therefore,

$$\sqrt{18+2\sqrt{14}+6\sqrt{2}+2\sqrt{63}}=\sqrt{2}+\sqrt{7}+\sqrt{9}.$$

No. 9. Proposed and solved by the Editor:

Solve for x:cos $[2\sin^{-1}|\tan(3 \text{ arc cot } x)|]=1$

Solution

$1-2\sin^2[\sin^{-1}|\tan(3 \text{ arc cot } x)|]=1$

$$\sin\left[\sin^{-1}\left\{\frac{3\tan(\text{arc cot } x - \tan^3(\text{arc cot } x)}{1-3\,[\tan^2(\text{arc cot } x)}\right\}\right.=0$$

$$\frac{\dfrac{3}{x}-\dfrac{1}{x^3}}{1-\dfrac{3}{x^2}}=0$$

$$\frac{3x^2-1}{x^3-3x}=0$$

$$x=\pm\,\infty \text{ or } \pm\frac{1}{\sqrt{3}}$$

Also solved by Chester R. Hillard, Wheaton College, Wheaton, Ill.

It is incomparably more important that the student develop ability to read a text with little or no aid from another than that he should learn to take notes from the instructor's lecture and return them to him written up in elegant but unchanged form. It is infinitely more important that he acquire the habit of doing his own thinking where humanly possible than that he should become merely adept in absorbing the thought of someone else. If nature has denied him the possession of even the beginnings of the power of independent thought, he has no place in a school—certainly is out of place in a mathematics class and should be treated as a mental defective. If he is in possession of the basis of a thinking ability—like all living things, this ability must be amenable to a proper treatment for its growth and development, and, if there is one undisputed principle in every province of nature, it is: No living thing can grow without some form of self-exercise.